

# Functions

## Part Two

# Outline for Today

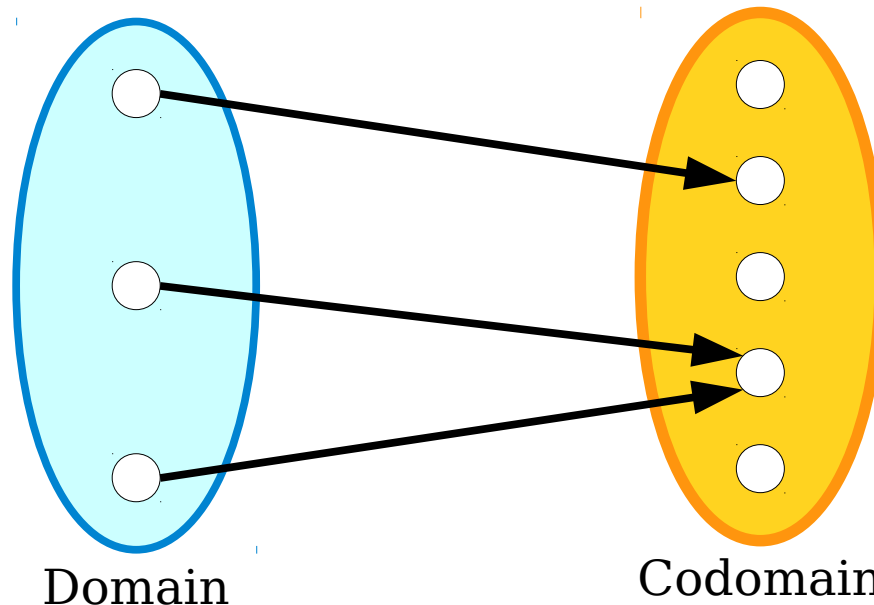
- ***Recap from Last Time***
  - Where are we, again?
- ***A Proof About Birds***
  - Trust me, it's relevant.
- ***Assuming vs Proving***
  - Two different roles to watch for.
- ***Connecting Function Types***
  - Relating the topics from last time.

Recap from Last Time

# Domains and Codomains

- Every function  $f$  has two sets associated with it: its **domain** and its **codomain**.
- A function  $f$  can only be applied to elements of its domain. For any  $x$  in the domain,  $f(x)$  belongs to the codomain.
- We write  $f : A \rightarrow B$  to indicate that  $f$  is a function whose domain is  $A$  and whose codomain is  $B$ .

The function must be defined for each element of its domain.



The output of the function must always be in the codomain, but not all elements of the codomain need to be producible.

# Involutions

- A function  $f : A \rightarrow A$  from a set back to itself is called an ***involution*** if the following first-order logic statement is true about  $f$ :

$$\forall x \in A. f(f(x)) = x.$$

*(“Applying  $f$  twice is equivalent to not applying  $f$  at all.”)*

- For example,  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined as  $f(x) = -x$  is an involution.

# Injective Functions

- A function  $f : A \rightarrow B$  is called **injective** (or **one-to-one**) if different inputs always map to different outputs.
  - A function with this property is called an **injection**.
- Formally,  $f : A \rightarrow B$  is an injection if this FOL statement is true:

$$\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2))$$

*(“If the inputs are different, the outputs are different”)*

- Equivalently:

$$\forall a_1 \in A. \forall a_2 \in A. (f(a_1) = f(a_2) \rightarrow a_1 = a_2)$$

*(“If the outputs are the same, the inputs are the same”)*

# Surjective Functions

- A function  $f : A \rightarrow B$  is called **surjective** (or **onto**) if this first-order logic statement is true about  $f$ :

$$\forall b \in B. \exists a \in A. f(a) = b$$

*(“For every output, there's an input that produces it.”)*

- A function with this property is called a **surjection**.

	To <i>prove</i> that this is true...	
$\forall x. A$	Have the reader pick an arbitrary $x$ . We then prove $A$ is true for that choice of $x$ .	
$\exists x. A$	Find an $x$ where $A$ is true. Then prove that $A$ is true for that specific choice of $x$ .	
$A \rightarrow B$	Assume $A$ is true, then prove $B$ is true.	
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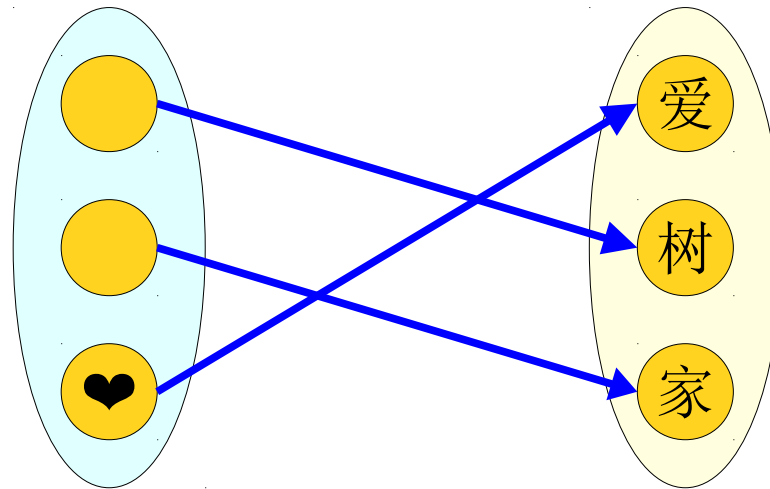
New Stuff!

# Injections and Surjections

- An injective function associates *at most* one element of the domain with each element of the codomain.
- A surjective function associates *at least* one element of the domain with each element of the codomain.
- What about functions that associate *exactly one* element of the domain with each element of the codomain?

# Bijections

- A ***bijection*** is a function that is both injective and surjective.
- Intuitively, if  $f : A \rightarrow B$  is a bijection, then  $f$  represents a way of pairing off elements of  $A$  and elements of  $B$ .



# Bijections

- Which of the following are bijections?
  - $f : \mathbb{R} \rightarrow \mathbb{R}$  defined as  $f(x) = x$ .
  - $f : \mathbb{Z} \rightarrow \mathbb{R}$  defined as  $f(x) = x$ .
  - $f : \mathbb{R} \rightarrow \mathbb{R}$  defined as  $f(x) = 2x + 1$ .
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# A Proof About Birds



***Theorem:*** If all birds can fly,  
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Given the predicates

*Bird*(*b*), which says *b* is a bird;

*Heron*(*h*), which says *h* is a heron; and

*CanFly*(*x*), which says *x* can fly,

translate the theorem into first-order logic.

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translate the theorem into first-order logic.

$(\forall b. (Bird(b) \rightarrow CanFly(b))) \rightarrow (\forall h. (Heron(h) \rightarrow CanFly(h)))$

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**Proof:** Assume that all birds can fly. We will show that all herons can fly.

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Answer at

<https://pollev.com/cs103>

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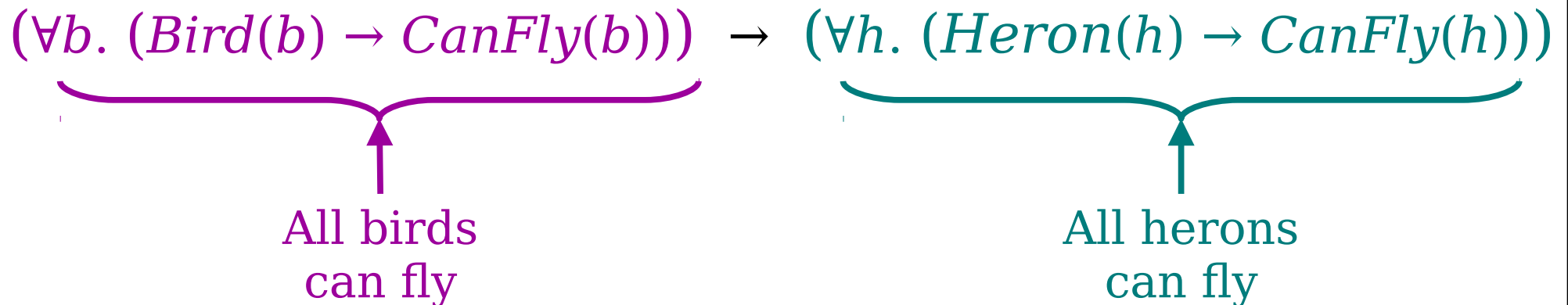
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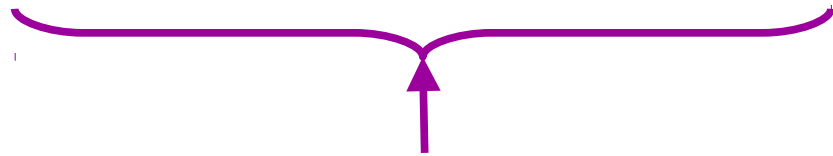


We introduce a variable  $h$  almost immediately.

# Proving vs. Assuming

- In the context of a proof, you will need to assume some statements and prove others.
  - Here, we **assumed** all birds can fly.
  - Here, we **proved** all herons can fly.
- Statements behave differently based on whether you're assuming or proving them.

$(\forall b. (Bird(b) \rightarrow CanFly(b))) \rightarrow (\forall h. (Heron(h) \rightarrow CanFly(h)))$



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# Proving vs. Assuming

- To **prove** the universally-quantified statement

$$\forall x. P(x)$$

we introduce a new variable  $x$  representing some arbitrarily-chosen value.

- Then, we prove that  $P(x)$  is true for that variable  $x$ .
- That's why we introduced a variable  $h$  in this proof representing a heron.

$$(\forall b. (Bird(b) \rightarrow CanFly(b))) \rightarrow (\forall h. (Heron(h) \rightarrow CanFly(h)))$$

We never introduce a variable  $b$ .

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# Proving vs. Assuming

- If we **assume** the statement

$$\forall x. P(x)$$

we **do not** introduce a variable  $x$ .

- Rather, if we find a relevant value  $z$  somewhere else in the proof, we can conclude that  $P(z)$  is true.
- That's why we didn't introduce a variable  $b$  in our proof, and why we concluded that  $h$ , our heron, can fly.

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We never introduce a variable  $b$ .

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$\exists x. A$	Find an $x$ where $A$ is true. Then prove that $A$ is true for that specific choice of $x$ .	Introduce a variable $x$ into your proof that has property $A$ .
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$A \vee B$	Either prove $\neg A \rightarrow B$ or prove $\neg B \rightarrow A$ . <i>(Why does this work?)</i>	Consider two cases. Case 1: $A$ is true. Case 2: $B$ is true.
$A \leftrightarrow B$	Prove $A \rightarrow B$ and $B \rightarrow A$ .	Assume $A \rightarrow B$ and $B \rightarrow A$ .
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

# Connecting Function Types

# Types of Functions

- We now have three special types of functions:
  - ***involutions***, functions that undo themselves;
  - ***injections***, functions where different inputs go to different outputs; and
  - ***surjections***, functions that cover their whole codomain.
- ***Question:*** How do these three classes of functions relate to one another?

***Theorem:*** For any function  $f : A \rightarrow A$ , if  $f$  is an involution, then  $f$  is surjective.

$$\underbrace{(\forall x \in A. f(f(x)) = x)}_{f \text{ is an involution.}} \rightarrow \underbrace{(\forall b \in A. \exists a \in A. f(a) = b)}_{f \text{ is surjective.}}$$

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$$(\forall x \in A. f(f(x)) = x) \rightarrow (\forall b \in A. \exists a \in A. f(a) = b)$$

Assume this.

Since we're assuming this, we aren't going to pick a specific choice of  $x$  right now. Instead, we're going to keep an eye out for something to apply this fact to.

Prove this.

### ***Proof Outline***

1. Assume  $f$  is an involution.

***Theorem:*** For any function  $f : A \rightarrow A$ , if  $f$  is an involution, then  $f$  is surjective.

$$(\forall x \in A. f(f(x)) = x) \rightarrow$$

$$(\forall b \in A. \exists a \in A. f(a) = b)$$

We've said that we need to prove this statement. How do we do that?

Prove this.

What do you do to prove  $\forall b \in A. [\text{something}]$ ?

Answer at  
<https://pollev.com/cs103>

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$$(\forall x \in A. f(f(x)) = x) \rightarrow (\forall b \in A. \exists a \in A. f(a) = b)$$

Assume this

To **prove** that  
this is true...

Have the reader pick an  
arbitrary  $x$ . Then prove  $A$  is  
true for that choice of  $x$ .

Prove this.

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Now, we hit an existential quantifier. Since we're proving this, we need to find a choice of  $a \in A$  where this is true.

Prove this.

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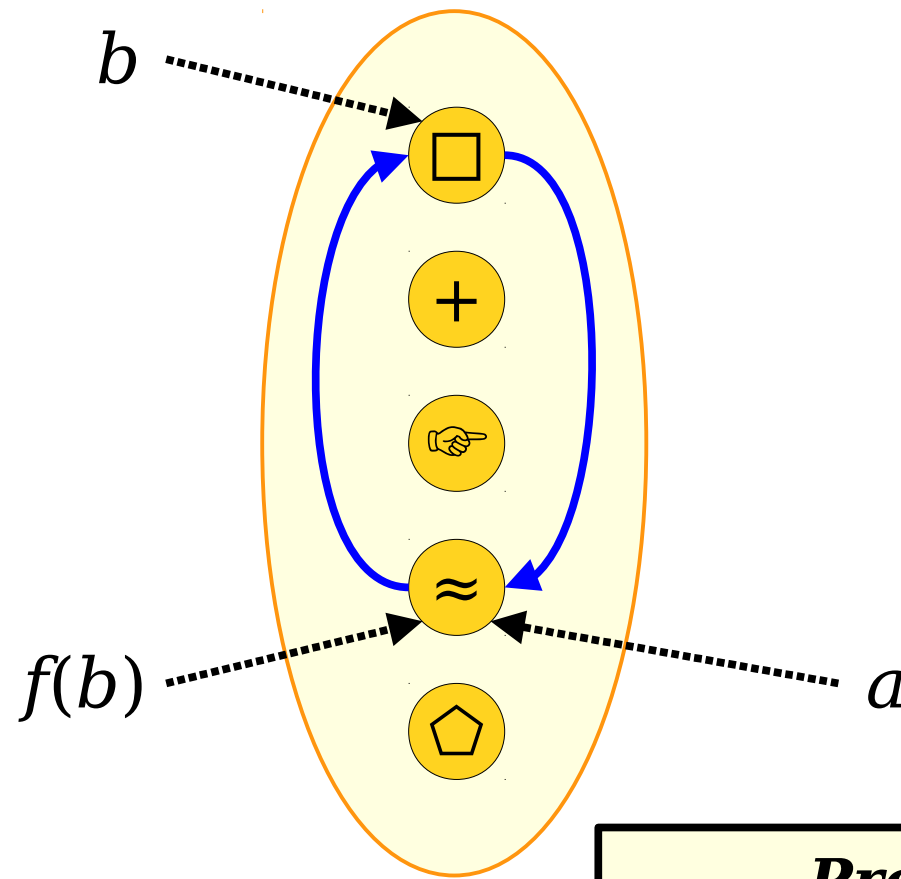
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Specifically, pick  $a = f(b)$ .

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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

# The Two-Column Proof Organizer

***Theorem:*** Let  $f : A \rightarrow A$  be an involution.  
Then  $f$  is injective.

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**What We're Assuming**

$f : A \rightarrow A$  is an involution.

$$\forall z \in A. f(f(z)) = z.$$

We're *assuming* this universally-quantified statement, so we won't introduce a variable for what's here.

**What We Need to Prove**

$f$  is injective.

$$\forall a_1 \in A. \forall a_2 \in A. (f(a_1) = f(a_2) \rightarrow a_1 = a_2)$$

We need to *prove* this universally-quantified statement. So let's introduce arbitrarily-chosen values.

**Theorem:** Let  $f : A \rightarrow A$  be an involution.  
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***What We're Assuming***

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$$\forall z \in A. f(f(z)) = z.$$

$$a_1 \in A$$

$$a_2 \in A$$

***What We Need to Prove***

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We need to prove this **implication**. So we **assume the antecedent** and **prove the consequent**.

**Theorem:** Let  $f : A \rightarrow A$  be an involution.  
Then  $f$  is injective.

**What We're Assuming**

$f : A \rightarrow A$  is an involution.

$$\forall z \in A. f(f(z)) = z.$$

$$a_1 \in A$$

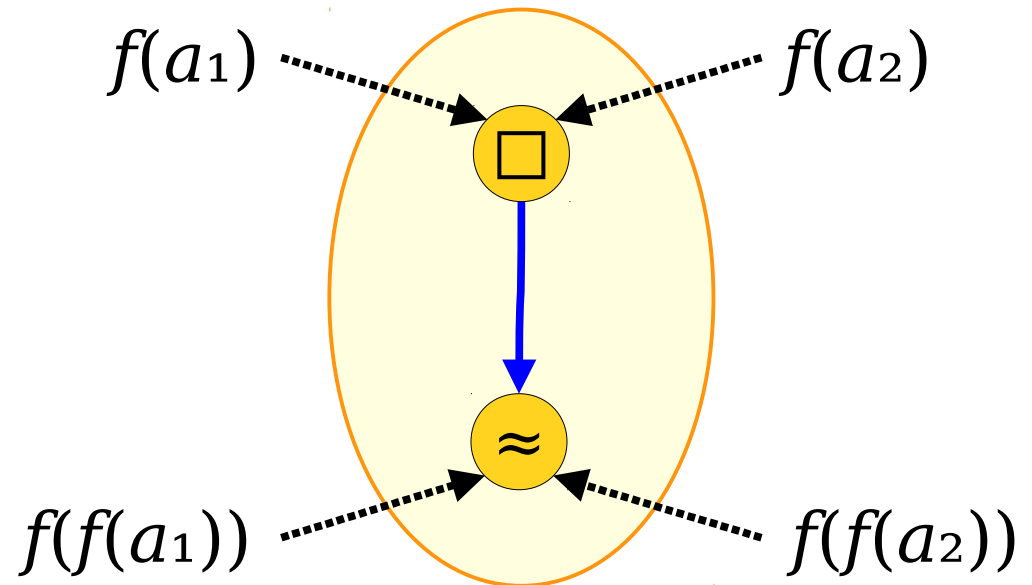
$$a_2 \in A$$

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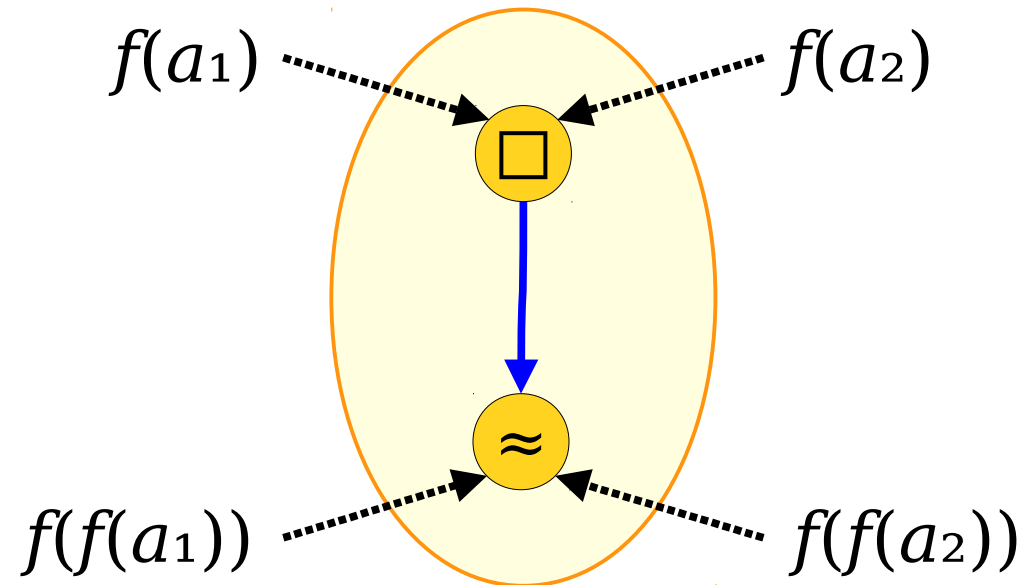
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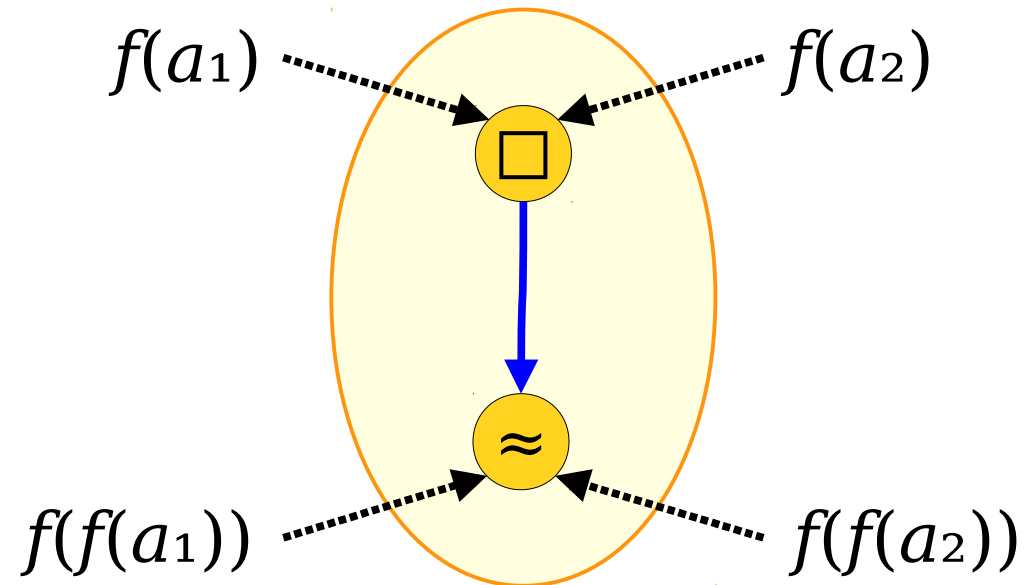
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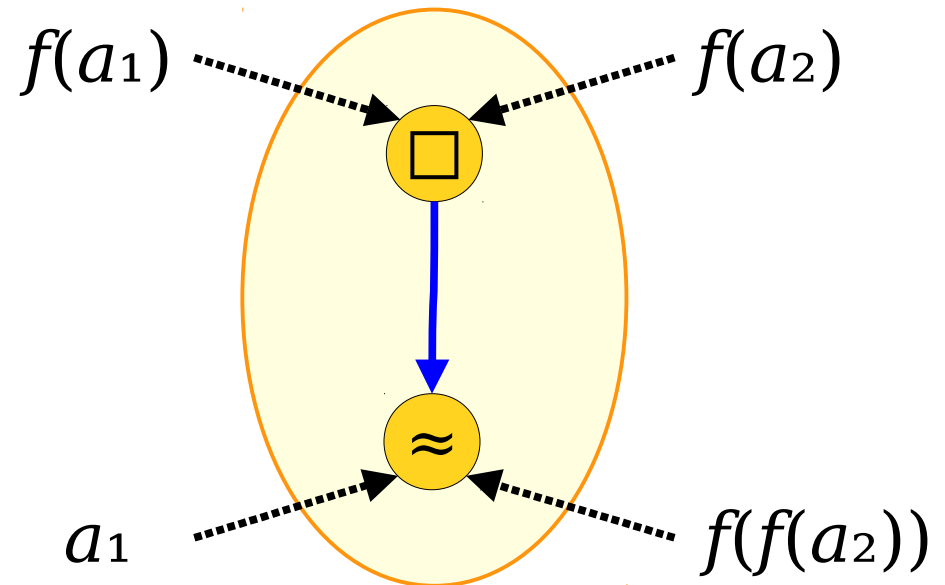
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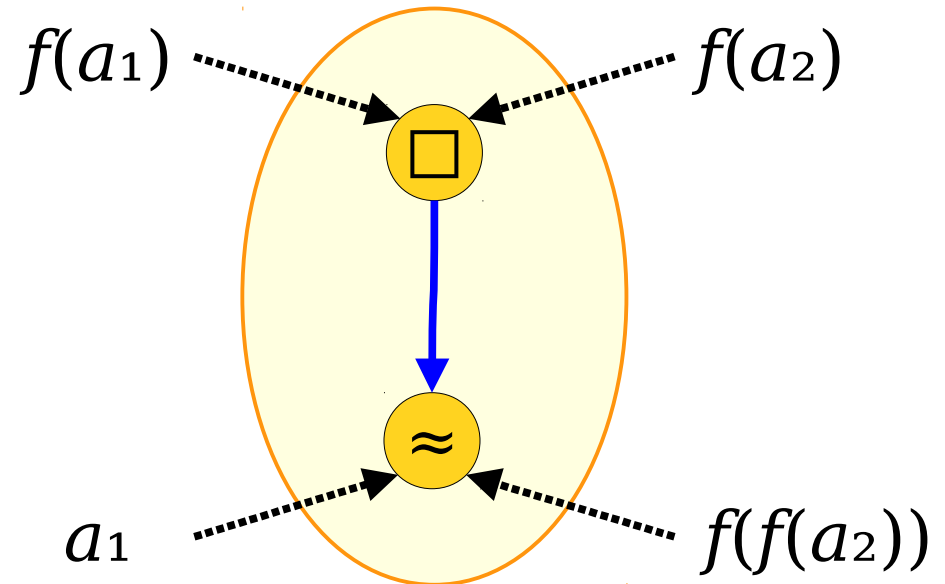
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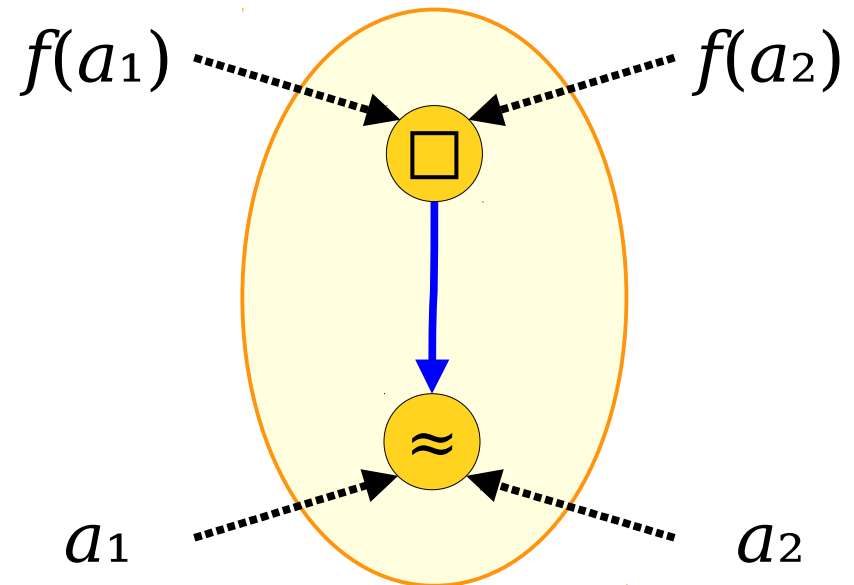
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$$a_1 = f(f(a_1)) = f(f(a_2)) = a_2,$$

so  $a_1 = a_2$ , as needed.

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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

**Time-Out for Announcements!**

# Problem Set Three

- PS2 was due today at 4:00PM.
  - Need more time? Use a late day to extend the deadline to 4:00PM tomorrow.
- PS3 goes out today. It's due next Friday at 4:00PM.
  - Explore properties of functions!
  - Prove results that have applications deep within CS theory and practice!
- As usual, feel free to stop by office hours or post on EdStem if you have any questions!

**Back to CS103!**

# Function Composition

***f : People → Places***

***g : Places → Prices***

Jennie

Cupertino, CA

Far Too Much

Wanyue

San Francisco

A King's Ransom

Vyoma

Redding, CA

A Modest Amount

Reva

Utqiagvik, AK

More Than  
You'd Expect

Tracy

Palo Alto, CA

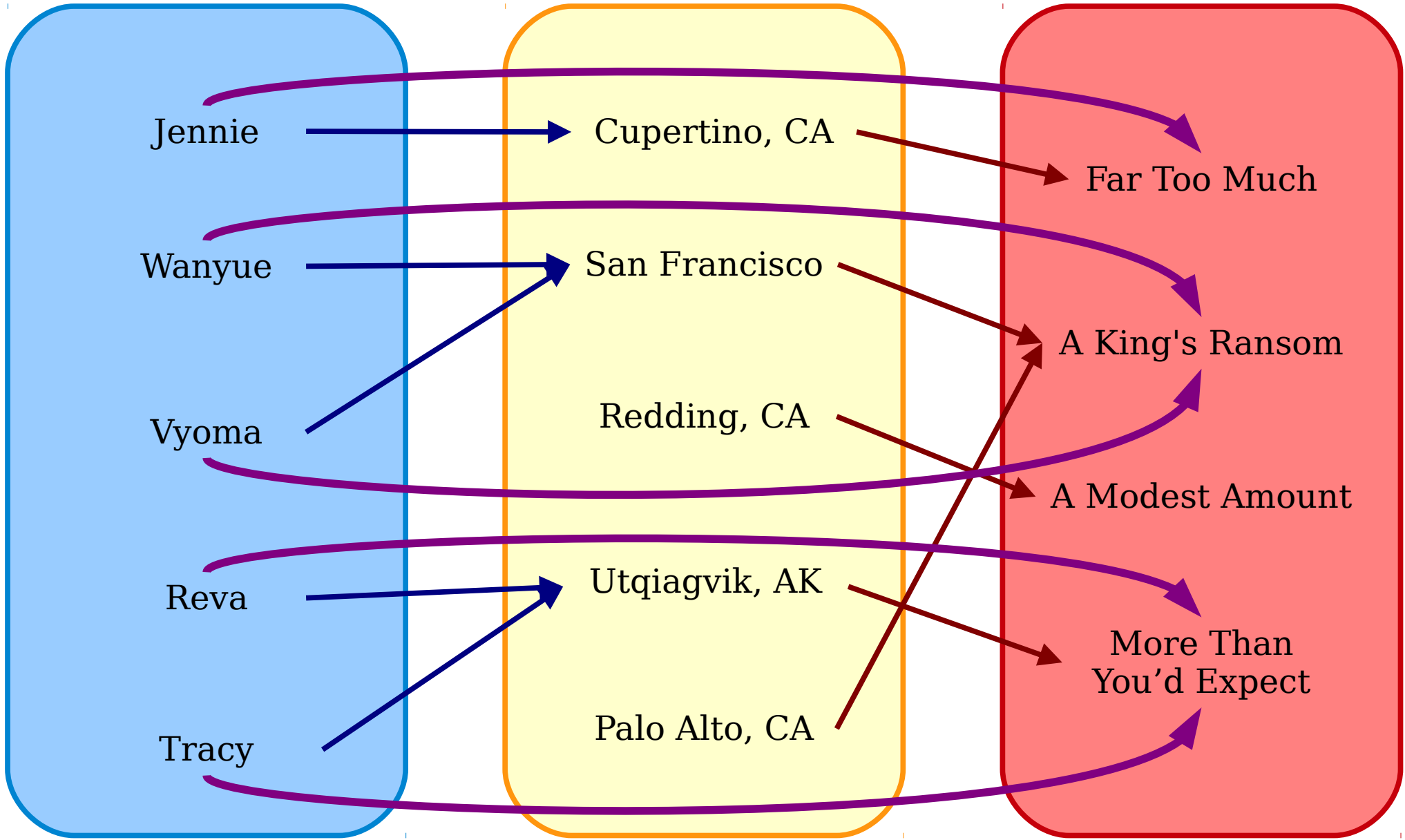
*People*

*Places*

*Prices*

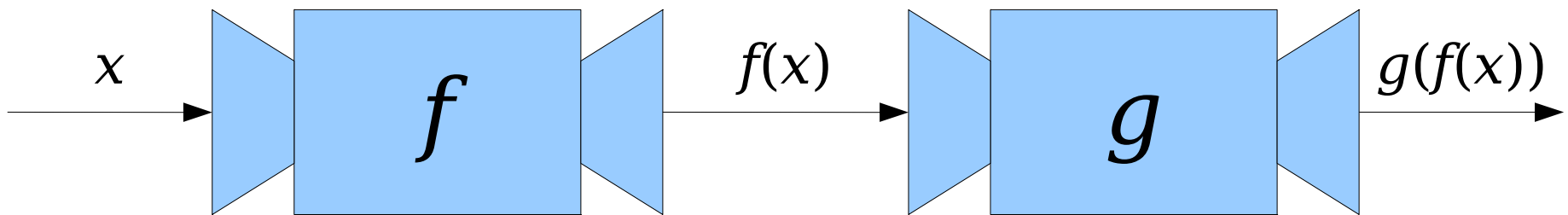
***h : People → Prices***

***h(x) = g(f(x))***



# Function Composition

- Suppose that we have two functions  $f : A \rightarrow B$  and  $g : B \rightarrow C$ .
- Notice that the codomain of  $f$  is the domain of  $g$ . This means that we can use outputs from  $f$  as inputs to  $g$ .



# Function Composition

- Suppose that we have two functions  $f : A \rightarrow B$  and  $g : B \rightarrow C$ .
- The **composition of  $f$  and  $g$** , denoted  $g \circ f$ , is a function where
  - $g \circ f : A \rightarrow C$ , and
  - $(g \circ f)(x) = g(f(x))$ .
- A few things to notice:
  - The domain of  $g \circ f$  is the domain of  $f$ . Its codomain is the codomain of  $g$ .
  - Even though the composition is written  $g \circ f$ , when evaluating  $(g \circ f)(x)$ , the function  $f$  is evaluated first.

The name of the function is  $g \circ f$ .  
When we apply it to an input  $x$ ,  
we write  $(g \circ f)(x)$ . I don't know  
why, but that's what we do.

# Properties of Composition

***Theorem:*** If  $f : A \rightarrow B$  is an injection and  $g : B \rightarrow C$  is an injection, then the function  $g \circ f : A \rightarrow C$  is an injection.

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### What We're Assuming

$f : A \rightarrow B$  is an injection.

$\forall x \in A. \forall y \in A. (x \neq y \rightarrow$   
 $f(x) \neq f(y))$

$g : B \rightarrow C$  is an injection.

$\forall x \in B. \forall y \in B. (x \neq y \rightarrow$   
 $g(x) \neq g(y))$

We're *assuming* these universally-quantified statements, so we won't introduce any variables for what's here.

### What We Need to Prove

$g \circ f$  is an injection.

$\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow$   
 $(g \circ f)(a_1) \neq (g \circ f)(a_2))$

We need to *prove* this universally-quantified statement. So let's introduce arbitrarily-chosen values.

**Theorem:** If  $f : A \rightarrow B$  is an injection and  $g : B \rightarrow C$  is an injection, then the function  $g \circ f : A \rightarrow C$  is an injection.

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$a_1 \in A$  is arbitrarily-chosen.

$a_2 \in A$  is arbitrarily-chosen.

**What We Need to Prove**

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$$\forall x \in A. \forall y \in A. (x \neq y \rightarrow f(x) \neq f(y))$$

$g : B \rightarrow C$  is an injection.

$$\forall x \in B. \forall y \in B. (x \neq y \rightarrow g(x) \neq g(y))$$

$a_1 \in A$  is arbitrarily-chosen.

$a_2 \in A$  is arbitrarily-chosen.

$a_1 \neq a_2$

### What We Need to Prove

$g \circ f$  is an injection.

$$\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow (g \circ f)(a_1) \neq (g \circ f)(a_2))$$

Now we're looking at an implication. Let's *assume* the antecedent and *prove* the consequent.

**Theorem:** If  $f : A \rightarrow B$  is an injection and  $g : B \rightarrow C$  is an injection, then the function  $g \circ f : A \rightarrow C$  is an injection.

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Let's write this out separately and simplify things a bit.

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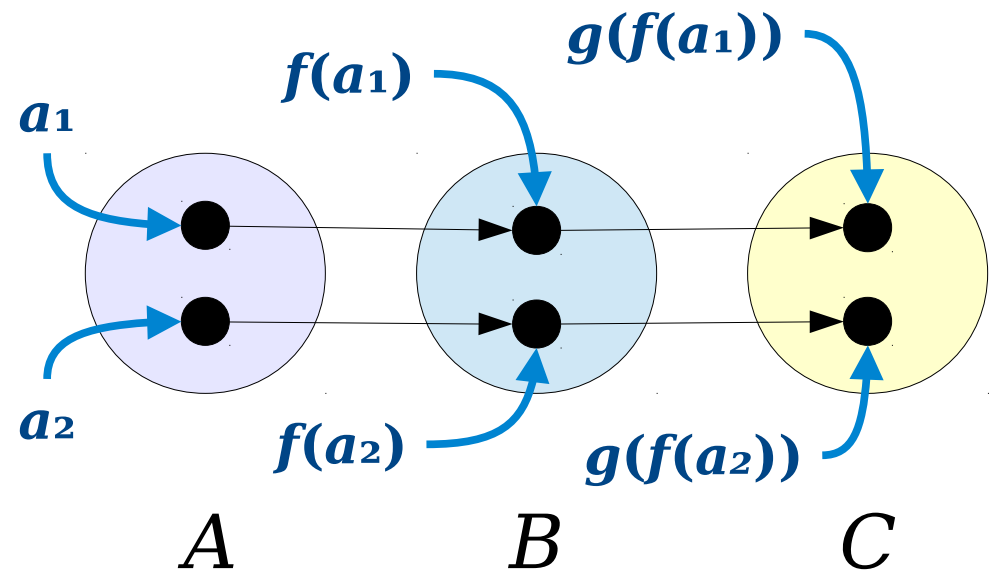
$$a_1 \neq a_2$$

**What We Need to Prove**

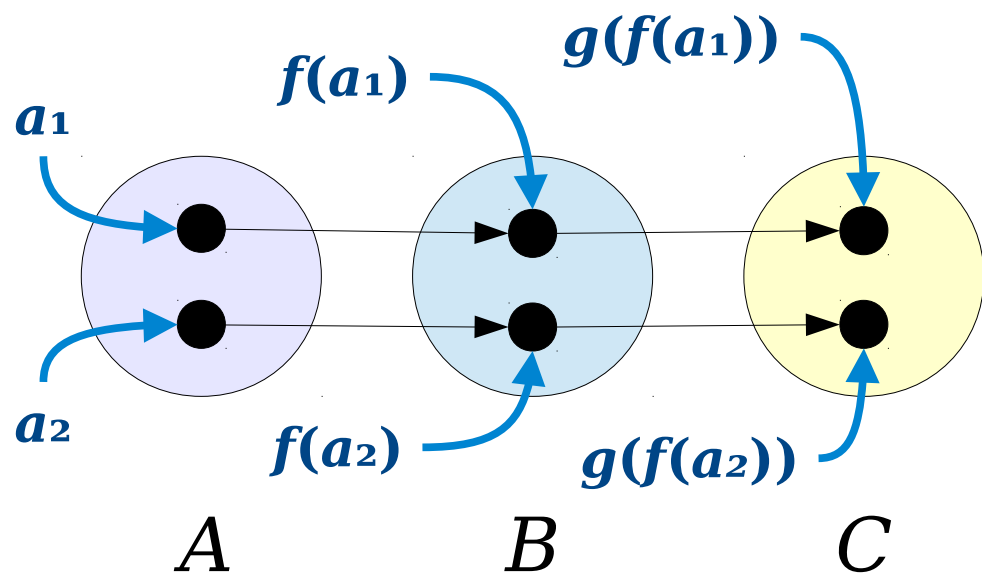
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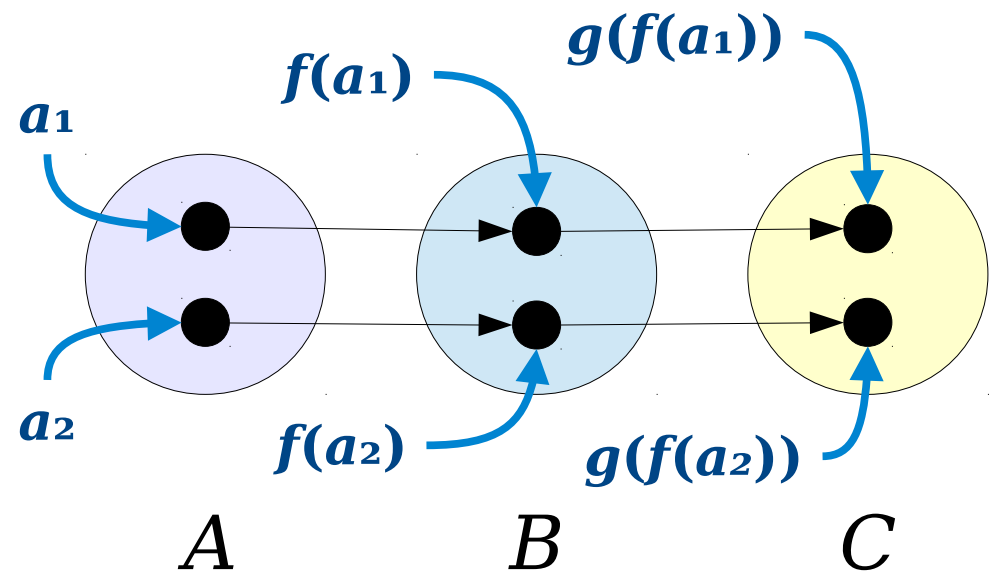


**Theorem:** If  $f : A \rightarrow B$  is an injection and  $g : B \rightarrow C$  is an injection, then the function  $g \circ f : A \rightarrow C$  is also an injection.



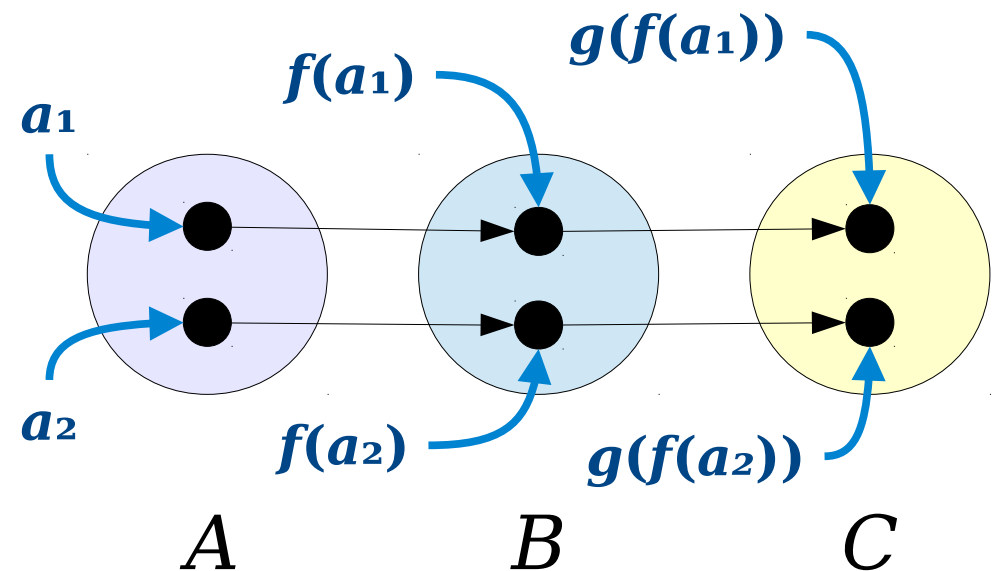
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**Proof:**



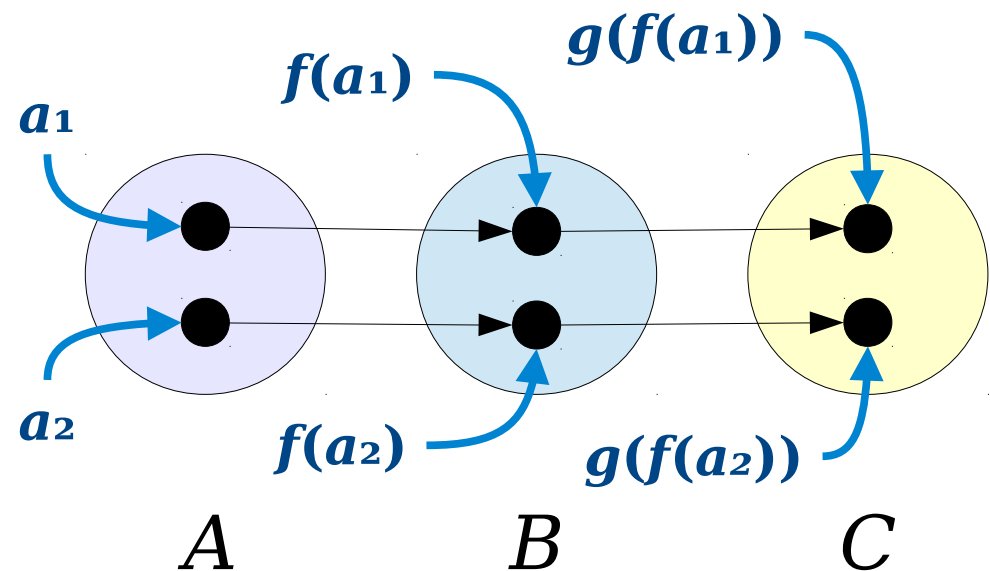
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**Proof:** Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be arbitrary injections.



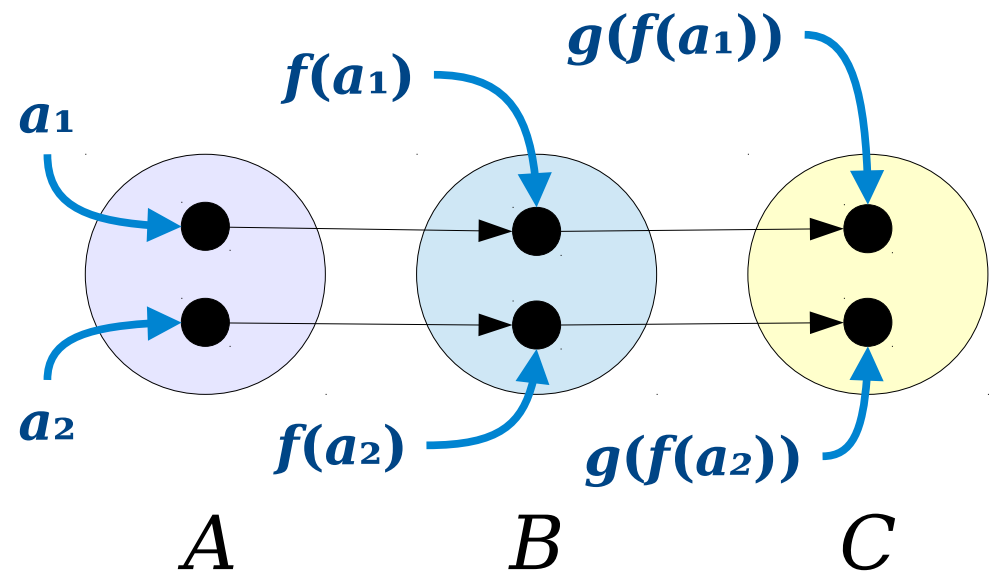
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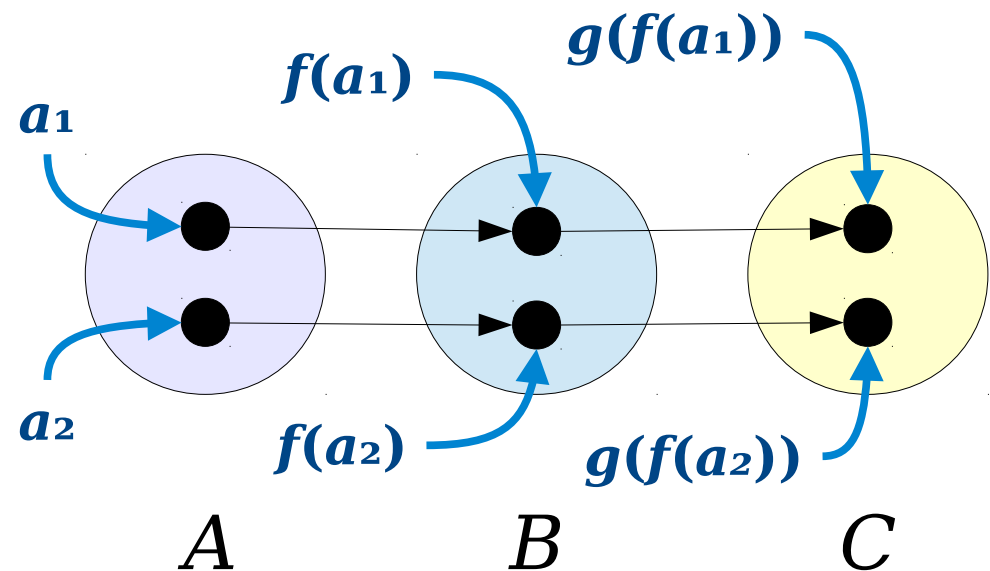
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**Proof:** Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be arbitrary injections. We will prove that the function  $g \circ f : A \rightarrow C$  is also injective. To do so, consider any  $a_1, a_2 \in A$  where  $a_1 \neq a_2$ .



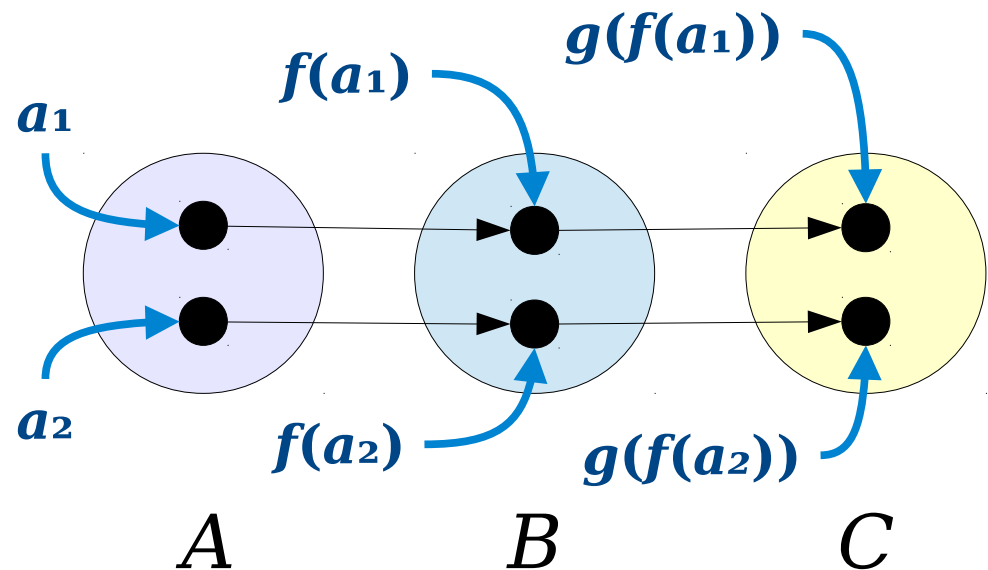
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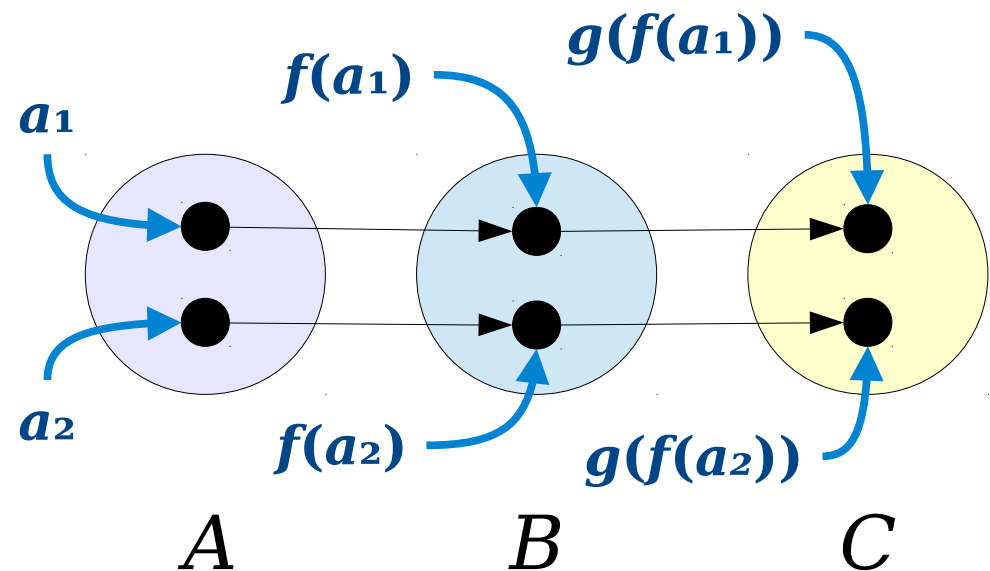
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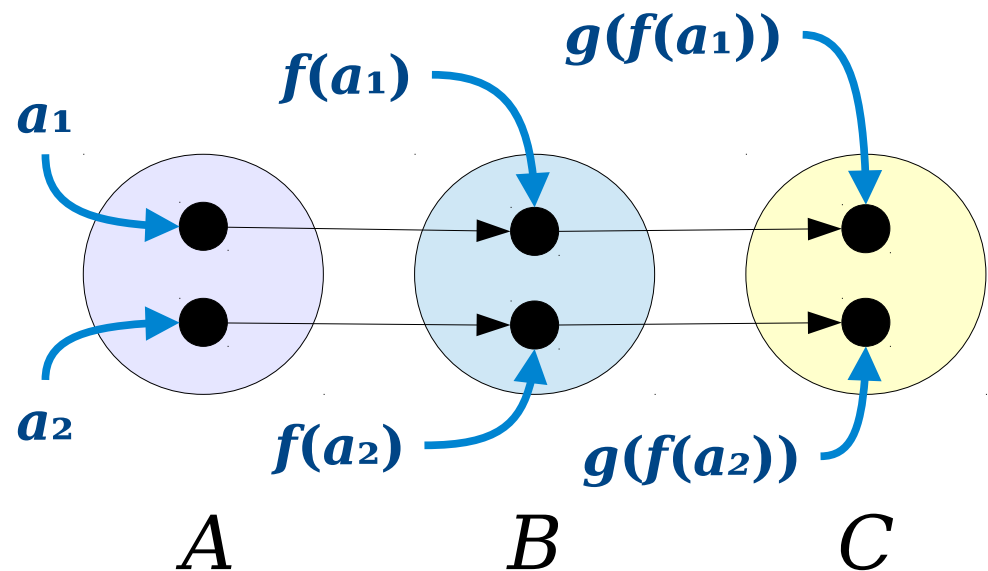
Since  $f$  is injective and  $a_1 \neq a_2$ , we see that  $f(a_1) \neq f(a_2)$ .



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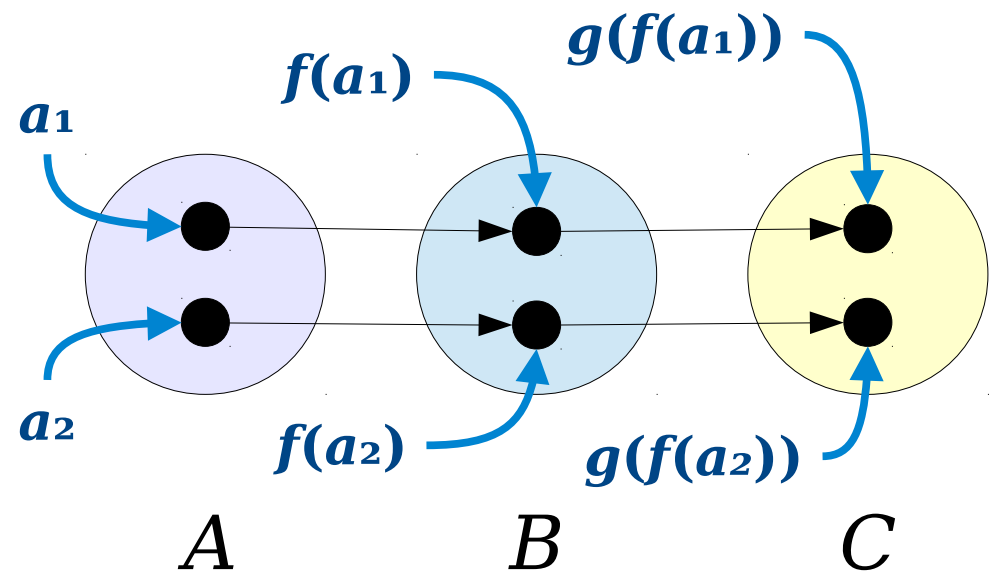
Since  $f$  is injective and  $a_1 \neq a_2$ , we see that  $f(a_1) \neq f(a_2)$ . Then, since  $g$  is injective and  $f(a_1) \neq f(a_2)$ , we see that  $g(f(a_1)) \neq g(f(a_2))$ , as required.



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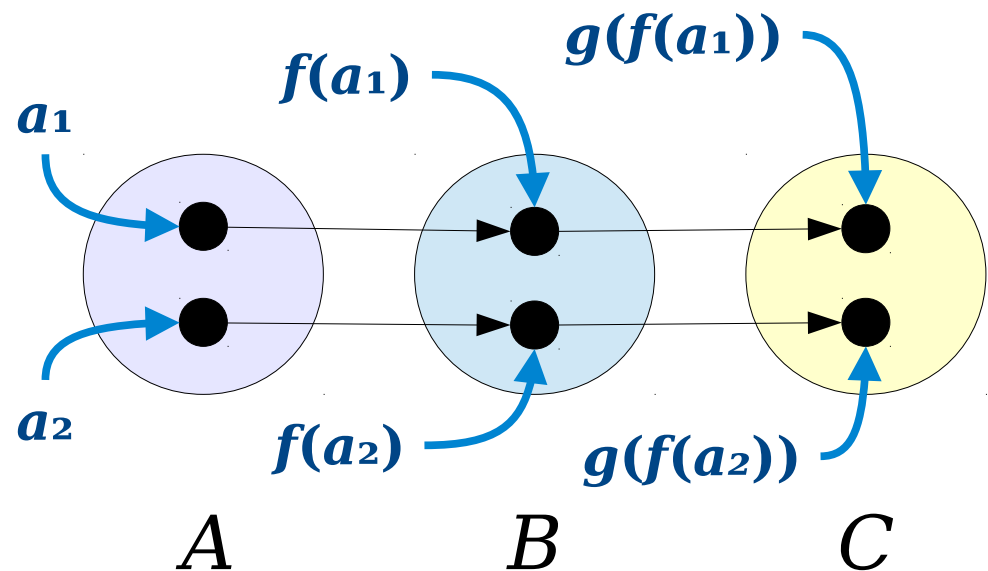


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Great exercise: Repeat this proof using the other definition of injectivity.

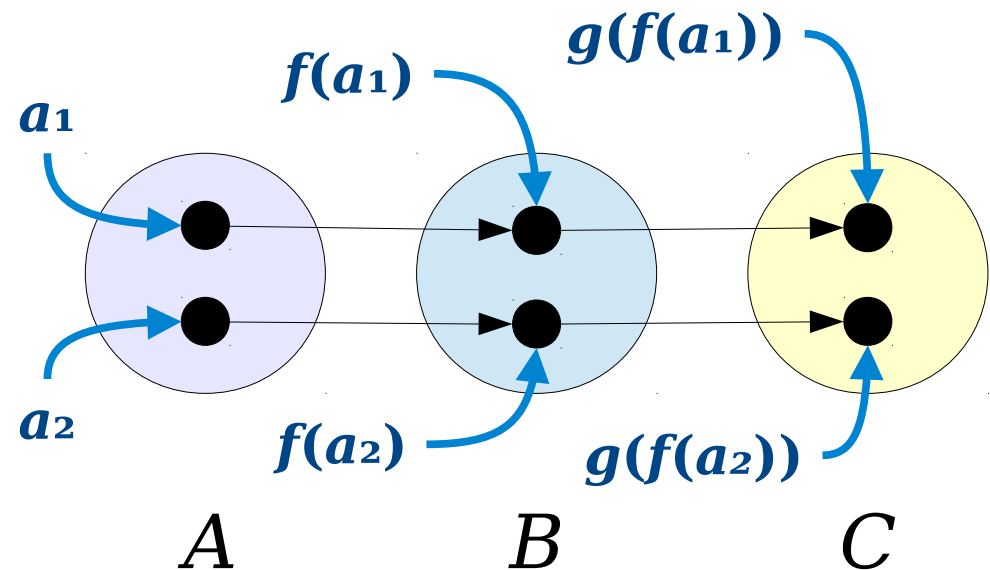


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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.



**Theorem:** If  $f : A \rightarrow B$  is a surjection and  $g : B \rightarrow C$  is a surjection, then the function  $g \circ f : A \rightarrow C$  is a surjection.

**Proof:** In the appendix!

# Major Ideas From Today

- Proofs involving first-order definitions are heavily based on the structure of those definitions, yet FOL notation itself does *not* appear in the proof.
- Statements behave differently based on whether you're **assuming** or **proving** them.
- When you **assume** a universally-quantified statement, initially, do nothing. Instead, keep an eye out for a place to apply the statement more specifically.
- When you **prove** a universally-quantified statement, pick an arbitrary value and try to prove it has the needed property.

	To <i>prove</i> that this is true...	If you <i>assume</i> this is true...
$\forall x. A$	Have the reader pick an arbitrary $x$ . We then prove $A$ is true for that choice of $x$ .	Initially, <b>do nothing</b> . Once you find a $z$ through other means, you can state it has property $A$ .
$\exists x. A$	Find an $x$ where $A$ is true. Then prove that $A$ is true for that specific choice of $x$ .	Introduce a variable $x$ into your proof that has property $A$ .
$A \rightarrow B$	Assume $A$ is true, then prove $B$ is true.	Initially, <b>do nothing</b> . Once you know $A$ is true, you can conclude $B$ is also true.
$A \wedge B$	Prove $A$ . Then prove $B$ .	Assume $A$ . Then assume $B$ .
$A \vee B$	Either prove $\neg A \rightarrow B$ or prove $\neg B \rightarrow A$ . <i>(Why does this work?)</i>	Consider two cases. Case 1: $A$ is true. Case 2: $B$ is true.
$A \leftrightarrow B$	Prove $A \rightarrow B$ and $B \rightarrow A$ .	Assume $A \rightarrow B$ and $B \rightarrow A$ .
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

# Next Time

- ***Graph Theory***

- A ubiquitous, powerful abstraction with applications throughout computer science.

- ***Vertex Covers***

- Making sure tourists don't get lost.

- ***Independent Sets***

- Helping the recovery of the California Condor.

## ***Appendix:*** Additional Function Proofs

***Proof:*** Composing surjections  
yields a surjection.

**Theorem:** If  $f : A \rightarrow B$  is surjective and  $g : B \rightarrow C$  is surjective, then  $g \circ f : A \rightarrow C$  is also surjective.

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Therefore, we'll choose an arbitrary  $c \in C$  and prove that there is some  $a \in A$  such that  $(g \circ f)(a) = c$ .

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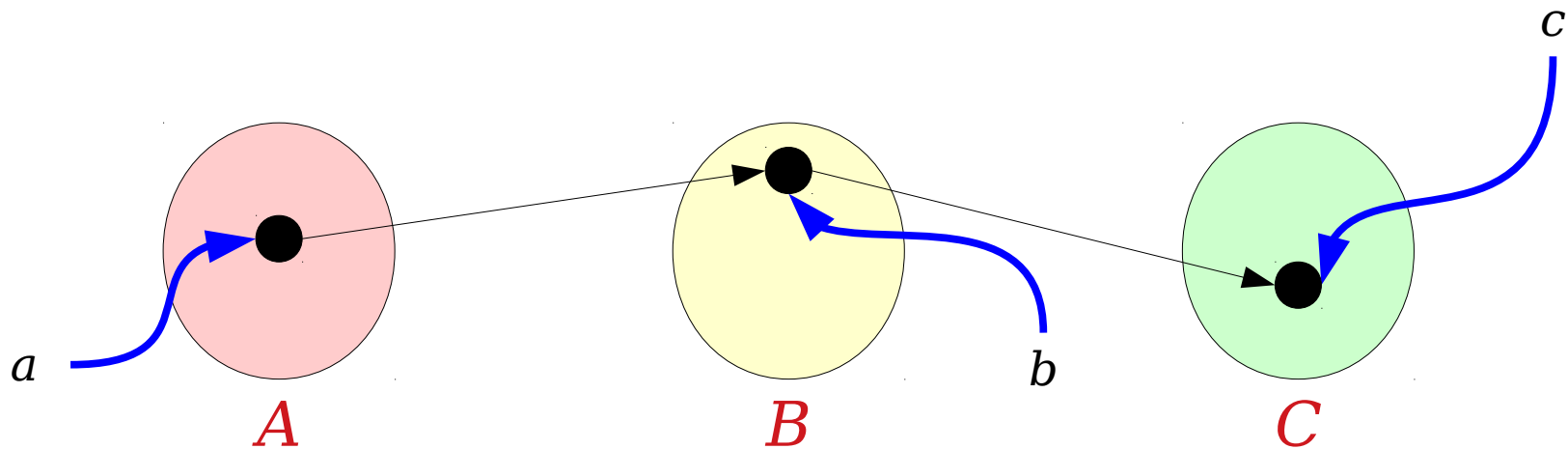
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**Proof:** Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be arbitrary surjections. We will prove that the function  $g \circ f : A \rightarrow C$  is also surjective. To do so, we will prove that for any  $c \in C$ , there is some  $a \in A$  such that  $(g \circ f)(a) = c$ .

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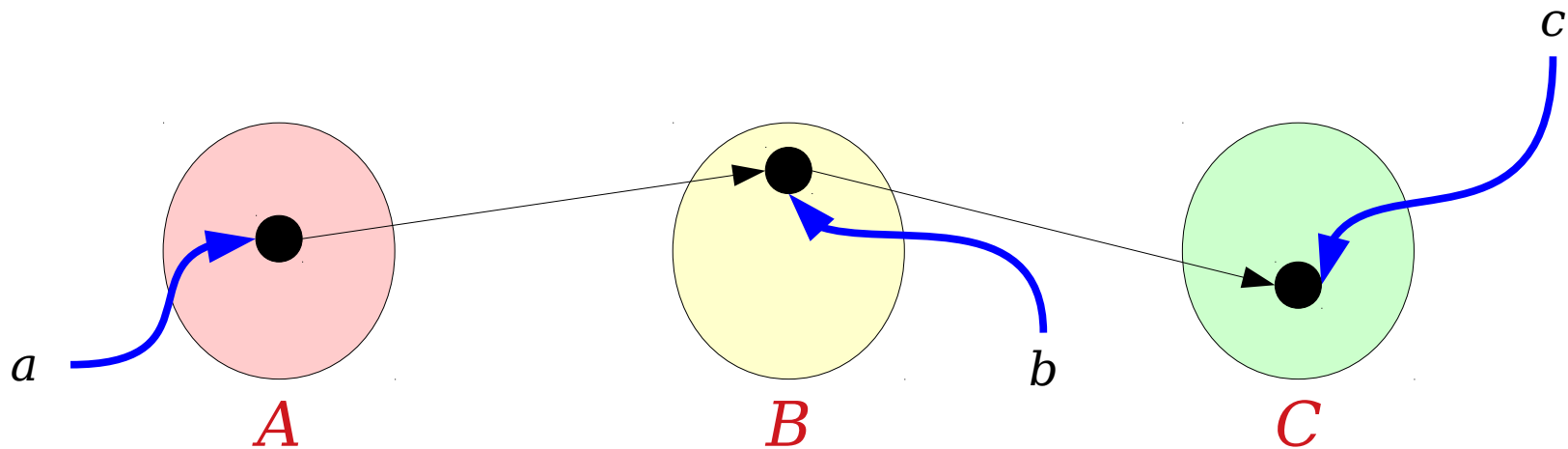
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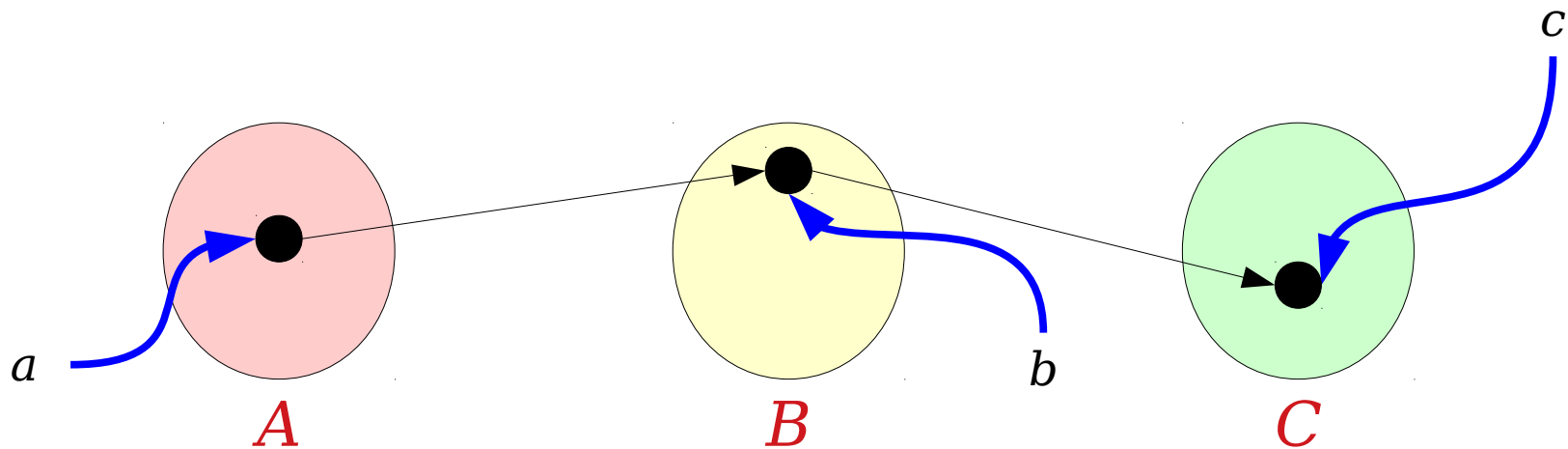
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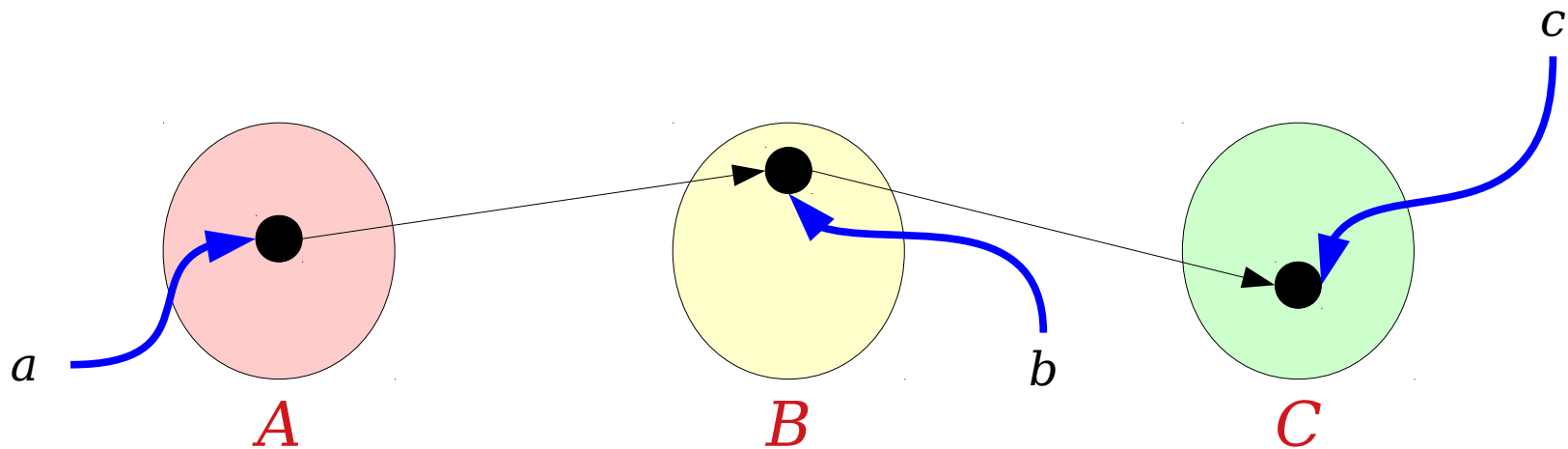
Consider any  $c \in C$ . Since  $g : B \rightarrow C$  is surjective, there is some  $b \in B$  such that  $g(b) = c$ .



**Theorem:** If  $f : A \rightarrow B$  is surjective and  $g : B \rightarrow C$  is surjective, then  $g \circ f : A \rightarrow C$  is also surjective.

**Proof:** Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be arbitrary surjections. We will prove that the function  $g \circ f : A \rightarrow C$  is also surjective. To do so, we will prove that for any  $c \in C$ , there is some  $a \in A$  such that  $(g \circ f)(a) = c$ . Equivalently, we will prove that for any  $c \in C$ , there is some  $a \in A$  such that  $g(f(a)) = c$ .

Consider any  $c \in C$ . Since  $g : B \rightarrow C$  is surjective, there is some  $b \in B$  such that  $g(b) = c$ . Similarly, since  $f : A \rightarrow B$  is surjective, there is some  $a \in A$  such that  $f(a) = b$ .



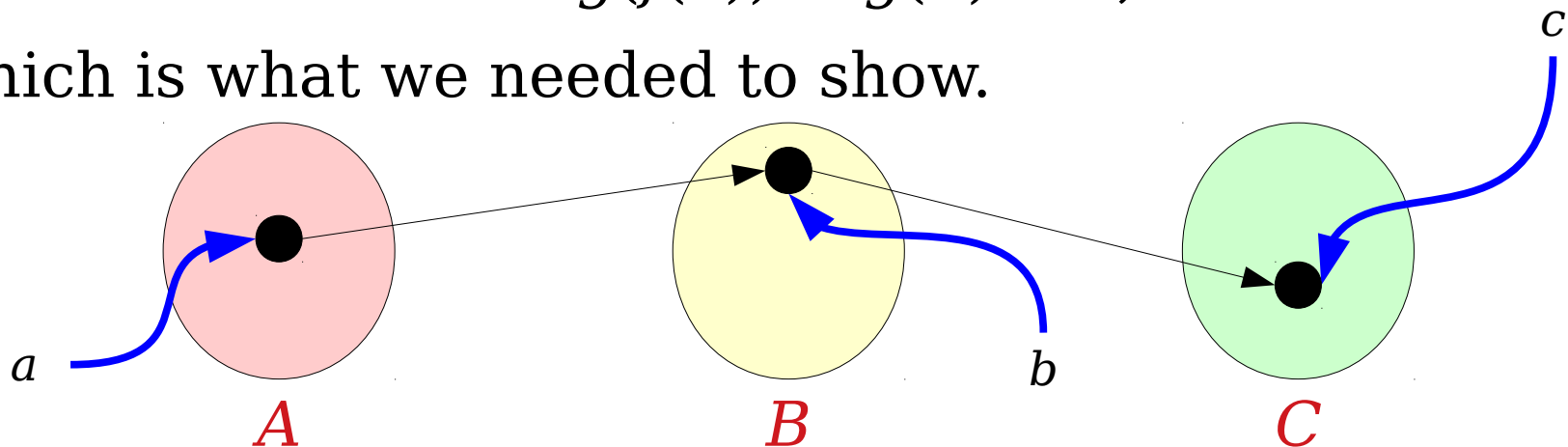
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